Quantisation of the non-linear super-Poincare fermion in a bag-like formulation

1979 J. Phys. A: Math. Gen. 12 2169

(http://iopscience.iop.org/0305-4470/12/11/026)

J. Phys. A: Math. Gen., Vol. 12, No. 11, 1979. Printed in Great Britain

Quantisation of the non-linear super-Poincaré fermion in a bag-like formulation

N S Baaklini and H B Bouzouita
International Centre for Theoretical Physics, PO Box 586–34100 Trieste, Italy

Received 19 January 1978

Abstract. The quantisation of the Volkov–Akulov fermion of non-linear super-Poincaré symmetry is studied using Dirac's generalised Hamiltonian method. The problem is found to be much simplified in a bag-like formulation of the model. The theory of a quark super-bag and a super-bubble is discussed.

1. Introduction

(b)

The Volkov–Akulov model (Volkov and Akulov 1973) of non-linear super-Poincaré symmetry (Golf'and and Likhtman 1971) describes a massless Goldstone fermion corresponding to the fermionic spinorial generator. The self-interactions of this fermion as well as its interactions with other fields are highly non-linear derivative expressions. The canonical quantisation of this theory, by Dirac's method (Dirac 1964), thus seems to be a formidable task.

Our purpose, in this paper, is two-fold. First, we shall simplify the problem of quantisation of the Volkov–Akulov non-linear fermion. This is achieved by reformulating the theory in a manner preserving the Goldstone vector field of four translations as a basic canonical variable. We recall that in the Volkov–Akulov approach, this field is identified with the Minkowskian space–time coordinate. In our approach, this field plays the role of the Minkowskian volume coordinate of a bag. However, after quantisation, we can identify this field with the parametrisation coordinates through gauge-fixing conditions.

Second, our purpose is to present this formulation of the Volkov-Akulov model of non-linear super-Poincaré symmetry as that of a quark-bag (or bubble) embedded in Minkowski space. Our work is thus a study of the canonical dynamics of this extended object. In § 2 we review the Volkov–Akulov model and present the problem of quantising the non-linear fermion. In § 3, we study the quantisation problem in a bag-like formulation. We consider the terms corresponding to the volume inside the bag. These are relevant to the Volkov-Akulov theory. In § 4, we consider the terms corresponding to the (membranous) bag surface. Finally, in § V, we discuss the relevance of our work to other more important problems.

2. The Volkov–Akulov non-linear fermion

Corresponding to the four-translational generator P_a and the (Majorana) fermionic supertranslational generator S_α of the super-Poincaré algebra, one introduces (Volkov and Akulov 1973, Volkov and Soroka 1974, Baaklini 1978, 1979), respectively, the Goldstone fields $v^a(x)$ and $\psi^\alpha(x)$. These transform as follows:

$$\delta v^a(x) = t^a + \bar{\epsilon} \mathrm{i} \gamma^a \psi(x),$$
$$\delta \psi^\alpha(x) = \epsilon^\alpha. \tag{1}$$

Here t^a and ϵ^α are constant (x-independent) parameters of four-translations and super-translations, respectively. Under the above transformations (1), the following one-forms are invariant:

$$\nabla v^a = \mathrm{d} v^a - \bar{\psi} \mathrm{i} \gamma^a \, \mathrm{d}\psi. \tag{2}$$

From these invariant one-forms, one constructs the invariant action

$$A = \int \tfrac{1}{24} \epsilon_{abcd} \nabla v^a \times \nabla v^b \times \nabla v^c \times \nabla v^d$$
$$= \int \mathrm{d}^4 x \, \tfrac{1}{24} \epsilon^{\mu\nu\lambda\rho} \epsilon_{abcd} \nabla_\mu v^a \nabla_\nu v^b \nabla_\lambda v^c \nabla_\rho v^d. \tag{3}$$

In the Volkov-Alkulov approach (Volkov and Akulov 1973), one identifies v^a with the Minkowskian space–time coordinate x_μ. Thus

$$\partial_\mu v^a = \delta^a_\mu. \tag{4}$$

The transformations (1) become

$$\delta x_\mu = t_\mu + \bar{\epsilon} \mathrm{i} \gamma_\mu \psi(x),$$
$$\delta \psi^\alpha(x) = \epsilon^\alpha + \bar{\epsilon} \mathrm{i} \gamma^\mu \psi \, \partial_\mu \psi + t^\mu \, \partial_\mu \psi. \tag{5}$$

The action (3) reduces to

$$A = \int \mathrm{d}^4 x \, [-1 + \bar{\psi} \mathrm{i} \not{\partial} \psi + \tfrac{1}{2}(T^\mu_\mu T^\nu_\nu - T^\nu_\mu T^\mu_\nu)$$
$$+ \tfrac{1}{6} \epsilon^{\mu\nu\lambda\rho} \epsilon_{\mu\sigma\tau\omega} T^\sigma_\nu T^\tau_\lambda T^\omega_\rho + \tfrac{1}{24} \epsilon^{\mu\nu\lambda\rho} \epsilon_{\sigma\tau\omega\kappa} T^\sigma_\mu T^\tau_\nu T^\omega_\lambda T^\kappa_\rho], \tag{6}$$

where

$$T_{\mu\nu} = \bar{\psi} \mathrm{i} \gamma_\mu \, \partial_\nu \psi. \tag{7}$$

The canonical momentum $\bar{\eta}^\alpha(x)$, conjugate to $\psi_\alpha(x)$, is

$$\bar{\eta}^\alpha = -\tfrac{1}{24} [\epsilon^{ijk} \epsilon_{\sigma\tau\omega\kappa} \bar{\psi} \mathrm{i} \gamma^\sigma (\delta^\tau_i - \bar{\psi} \mathrm{i} \gamma^\tau \, \partial_i \psi)(\delta^\omega_j - \bar{\psi} \mathrm{i} \gamma^\omega \, \partial_j \psi)(\delta^\kappa_k - \bar{\psi} \mathrm{i} \gamma^\kappa \, \partial_k \psi)]^\alpha. \tag{8}$$

From (8), we have the weakly vanishing (≈ 0) second-class constraint

$$\bar{\kappa}^\alpha(x) \equiv \bar{\eta}^\alpha(x) + \tfrac{1}{24} [\epsilon^{ijk} \epsilon_{\sigma\tau\omega\kappa} \bar{\psi} \mathrm{i} \gamma^\sigma (\delta^\tau_i - \bar{\psi} \mathrm{i} \gamma^\tau \, \partial_i \psi)$$
$$\times (\delta^\omega_j - \bar{\psi} \mathrm{i} \gamma^\omega \, \partial_j \psi)(\delta^\kappa_k - \bar{\psi} \mathrm{i} \gamma^\kappa \, \partial_k \psi)]^\alpha \approx 0. \tag{9}$$

In order to put the constraint (9) strongly equal to zero, thus eliminating $\bar{\eta}^\alpha$ in Dirac's method (Dirac 1964), one must define new Poisson (or Dirac) brackets,

$$\{f, g\}^* = \{f, g\} - \{f, \bar{\kappa}^\alpha\} M^{-1}_{\alpha\beta} \{\bar{\kappa}^\beta, g\}, \tag{10}$$

for any two functions f and g of the canonical variables. Here $M_{\alpha\beta}$ is defined by

$$\{\bar{\kappa}_\alpha, \bar{\kappa}_\beta\}_+ = M_{\alpha\beta}. \tag{11}$$

From (9) and (11) we thus see that the matrix $M_{\alpha\beta}$ is a complicated expression.

Consequently, its inverse and the quantisation rules based on the resulting Dirac brackets are very complicated. Hence we propose a different scheme based on the action (3), without identifying v^a with x_μ through the condition (4). In the following section, we consider both $v^a(x)$ and $\psi^\alpha(x)$ as the basic canonical variables in the process of quantisation.

3. Quantisation in a bag-like formulation

The Volkov–Akulov action (3) can be rewritten in the form

$$A = \int d^4x\,(-\det g_{\mu\nu})^{1/2}, \tag{12}$$

where

$$g_{\mu\nu} = \nabla_\mu v^a \nabla_\nu v_a. \tag{13}$$

The relation of the form (12) to the actions of the string, the membrane and the bag is very obvious. Actually, the form (12) is, in the absence of the fermion, the usual volume term of the bag action.

Besides invariance under (global) Lorentz transformations of the embedding Minkowski space, acting on the vector index a and the spinor index α, the action (12) is invariant under the (global) four-translational and the super-translational transformations (1). Moreover, it is invariant under general coordinate transformations or reparametrisations acting on the index μ.

The momenta $P^a(x)$ and $\bar\eta^\alpha(x)$, conjugate to $v^a(x)$ and $\psi^\alpha(x)$ are, respectively,

$$P^a = (-\det g_{\mu\nu})^{1/2}\, e^{\mu 0}\nabla_\mu v^a, \tag{14}$$

$$\bar\eta^\alpha = -P^a(\bar\psi \mathrm{i}\gamma_a)^\alpha, \tag{15}$$

where $e^{\mu\nu}$ is the inverse of $g_{\mu\nu}$,

$$e^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda. \tag{16}$$

We define the fundamental Poisson brackets:

$$\begin{aligned} &\{v_a(\boldsymbol{x}), P^b(\boldsymbol{y})\}_- = \delta^b_a \delta^3(\boldsymbol{x}-\boldsymbol{y}),\\ &\{\psi_\alpha(\boldsymbol{x}), \bar\eta^\beta(\boldsymbol{y})\}_+ = \delta^\beta_\alpha \delta^3(\boldsymbol{x}-\boldsymbol{y}). \end{aligned} \tag{17}$$

From (14) and (15) we obtain the constraints

$$\chi_i \equiv P^a \nabla_i v_a \approx 0, \qquad i = 1, 2, 3, \tag{18}$$

$$\chi_0 \equiv P_a^2 + \det|g_{ij}| \approx 0, \tag{19}$$

$$\bar\kappa^\alpha \equiv \bar\eta^\alpha + P^a(\bar\psi \mathrm{i}\gamma_a)^\alpha \approx 0. \tag{20}$$

Using (17)–(20), we find that the Poisson brackets of χ_i and χ_0 among themselves and with $\bar K_\alpha$ are weakly vanishing. Moreover, we have

$$\begin{aligned} \{\bar\kappa_\alpha(\boldsymbol{x}), \bar\kappa_\beta(\boldsymbol{y})\}_+ &= -2P^a(c^{-1}\mathrm{i}\gamma_a)_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y})\\ &\equiv M_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{y}). \end{aligned} \tag{21}$$

The Hamiltonian of the theory is vanishing up to the contraints (18) and (19). Hence χ_i

and χ_0 are first-class constraints corresponding to the invariance under four-reparametrisations. The constraints $\bar{K}_\alpha$ are second class. Note that, contrary to the case of the previous section, the matrix $M_{\alpha\beta}$ defined by (21) is simple. Consequently one easily calculates the Dirac brackets of equations (10).

The Dirac brackets for our canonical variables are

$$\begin{aligned}
&\{v^a(\boldsymbol{x}), v^b(\boldsymbol{y})\}^*_- = \tfrac{1}{2}\epsilon^{abcd}(P_c/P^2)\bar{\psi}(\boldsymbol{x})i\gamma_d\gamma_5\psi(\boldsymbol{y}),\\
&\{v^a(\boldsymbol{x}), \psi_\alpha(\boldsymbol{y})\}^*_- = -\tfrac{1}{2}(\not{P}\gamma^a\psi/P^2)\alpha\delta^3(\boldsymbol{x}-\boldsymbol{y}),\\
&\{\psi_\alpha(\boldsymbol{x}), \psi_\beta(\boldsymbol{y})\}^*_+ = \tfrac{1}{2}(\mathrm{i}\not{P}C/P^2)\alpha\beta\delta^3(\boldsymbol{x}-\boldsymbol{y}).
\end{aligned} \tag{22}$$

In order to put the four first-class constraints χ_i and χ_0 strongly equal to zero, we must introduce four corresponding gauge-fixing conditions. We may take the constraints

$$c_\mu \equiv v_\mu - x_\mu \approx 0. \tag{23}$$

The gauge-fixing constraints (23) have the effect, like the Volkov–Akulov restriction (4), of identifying, inside the bag volume, the Minkowskian bag coordinates with the proper parametrisation coordinates.

Nevertheless, one may consider the constraints χ_i and χ_0 as functional differential equations obeyed by the state functional $\phi(v^a, \psi^\alpha)$, by making the usual substitution, $P_a \to \delta/\mathrm{i}\delta v^a$. The functional differential equation

$$\left(-\frac{\delta^2}{\delta v_a^2} + \det|g_{ij}|\right)\phi(v^a, \psi^\alpha) = 0, \tag{24}$$

corresponding to χ_0, is like the Klein–Gordon equation for a point particle. The role played by the anticommuting quantum field variables $\psi^\alpha(\boldsymbol{x})$ in the above equation is not clear. It is by solving (24) and also the equations corresponding to χ_i, that one finds the quantum state inside the bag volume, and, similarly, the quantum states of the Volkov–Akulov theory.

4. The dynamics of a super-bubble

The conventional bag model of hadrons (Hasenfratz and Kuti 1978) contains a surface term (a bubble or a membrane), besides the volume term. We can similarly introduce such a term

$$A = \sigma \int_{x_1=1} \mathrm{d}x^0\, \mathrm{d}x^2\, \mathrm{d}x^3\, (\det g_{rs})^{1/2}, \tag{25}$$

$$g_{rs} = \nabla_r v^a \nabla_s v_a; \qquad r, s = 0, 2, 3. \tag{26}$$

Here σ is the surface tension and $x_1 = 1$ defines the bag surface.

The study of the dynamics of this surface term is very similar to that of the previous section. However, whereas the Minkowskian volume coordinates can all be identified (via gauge-fixing) with the parametrisation coordinates, three of the Minkowskian surface coordinates can be identified in this way and the remaining one is left as a free canonical quantum variable.

Proceeding as in the previous section, we define the (surface) momenta, conjugate to v^a and ψ^α respectively by

$$\pi^a = (\det g_{rs})^{1/2} f^{t0} \nabla_t v^a, \tag{27}$$

$$\bar{\xi}^\alpha = -\pi^a (\bar{\psi} \mathrm{i} \gamma_a)^\alpha, \tag{28}$$

where f^{rs} is the inverse of g_{rs}.

Thus we have the first-class constraints

$$\Sigma_R \equiv \pi^a \nabla_R v_a \approx 0, \qquad R = 2, 3, \tag{29}$$

$$\Sigma_0 \equiv \pi_a^2 + \det|g_{RS}| \approx 0, \tag{30}$$

and the fermion second-class constraint

$$\bar{\Lambda}^\alpha \equiv \bar{\xi}^\alpha + \pi^a (\bar{\psi} \mathrm{i} \gamma_a)^\alpha \approx 0. \tag{31}$$

The elimination of the fermion constraint is similar to the previous section. The gauge-fixing conditions corresponding to Σ_R and Σ_0 can be taken to be

$$c_r \equiv v_r - x_r \approx 0. \tag{32}$$

Therefore, $v_1(x)$ is left, together with $\psi_\alpha(x)$ as a canonical variable.

On the basis of the boundary conditions determined by the geometry of the bubble, one expands the canonical variables in terms of the Fourier series and proceeds to obtain the spectrum of the quantum states. This work will be relegated to a more realistic and physically interesting model of which the present one is only a part.

5. Discussion

In this paper we have pointed out the simplification encountered in the formulation of the problem of the quantisation of the Volkov–Akulov super-Poincaré fermion. This is achieved in a bag-like formulation of the model. Such a result seems to indicate the relevance of (non-linear) supersymmetry to the physics of extended objects. In the locally super-Poincaré symmetric extension of the theory (Volkov and Soroka 1974, Baaklini 1978, 1979) one introduces spin-2 and spin-$\frac{3}{2}$ fields as gauge fields in the bag. It has been proposed (Baaklini 1978, 1979) that such a bag model of supergravity may serve as a geometric theory of hadron structure. This is our basic physical motivation for the present formulation of the quantisation problem.

On the other hand, leaving the application to hadronic physics aside, a locally Poincaré (and super-Poincaré) symmetric bag-like model of gravity (and supergravity) is an extension which deserves particular attention in itself, mainly in relation to the quantisation problem of these theories.

6. Conclusion

We have considered in the present formulation of the quantisation problem the translational Goldstone fields of the global super-Poincaré symmetry. A study of the spectrum of the quantum states, i.e. the solution of the quantum problem, would be more appropriate in the physically more important theory which incorporates the local symmetries.

Acknowledgments

The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

References

Baaklini N S 1978 *J. Phys. G: Nucl. Phys.* **4** 1
—— 1979 *Hadronic J.* **2** 135
Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University)
Golf'and Y A and Likhtman E P 1971 *Zh. Eksp. Teor. Fiz. Pis. Red.* **13** 452
Hasenfratz P and Kuti J 1978 *Phys. Rep.* **40** No 2
Volkov D V and Akulov V P 1973 *Phys. Lett. B* **46** 109
Volkov D V and Soroka V A 1974 *Theor. Math. Phys* **20** 829